Bound states and resonances produced by a sum of separable potentials

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Trevor W Preist

Department of Physics, University of Exeter, Stocker Road, Exeter EX4 4QL, UK

Abstract. It is shown how an arbitrary number of bound states and resonances with prescribed energies and form factors may be derived from an equivalent separable potential. It is then possible to express the two-body T matrix exactly in terms of the bound state and resonance parameters without explicitly introducing the potential.

1. Introduction

The study of the three-body problem using Faddeev methods (Faddeev 1961a, b) has provided a stimulus for investigating the off-shell behaviour of two-body T-matrices (Lovelace 1964a, b, Kowalski 1965, Brayshaw 1968a, b, Mongan 1969). Realistic calculations of three-body systems require reliable two-body information and unfortunately this cannot always be constructed in a convenient form.

It is well known that the physical or on-shell behaviour of the phase shift does not uniquely determine the off-shell behaviour. The usual procedure is to choose a physically reasonable class of potentials with parameters adjusted to give the appropriate physics. Having done this the potential may be used to predict the off-shell behaviour although this is not often possible analytically. Lovelace (1964b) gave a number of ways of constructing the T matrix describing a system with a single bound state or resonance. One of these, based upon the use of a separable potential, results in a form particularly useful for three-body calculations and in this paper we develop a prescription which allows the inclusion of an arbitrary number of bound states and resonances with known wavefunctions.

2. Separable potentials

In a single partial wave the T matrix equation is (Lovelace 1964b)

$$\hat{T}_l = \hat{V}_l - \hat{V}_l \hat{G}_0^l \hat{T}_l \tag{1}$$

with

$$\langle p | \hat{G}_0^l | p' \rangle = \frac{1}{4\pi p^2} \frac{\delta(p - p')}{p^2 - s},$$

s being the energy variable.

For a simple separable potential

$$\langle p|\hat{V}_l|p'\rangle = \lambda_l g_l(p)g_l(p') \tag{2}$$

and $g_l(p)$ is the real potential form factor.

The corresponding T matrix is readily found to be (Yamaguchi 1954)

$$T_l(p, p', s) = g_l(p)\tau_l(s)g_l(p') \tag{3}$$

with

$$\tau_l(s)^{-1} = \frac{1}{\lambda_l} + 4\pi \int_0^\infty \frac{|g_l(q)|^2 q^2 \, dq}{q^2 - s}.$$

This separable potential will produce a bound state at $s = -E_\mu$ if

$$-\frac{1}{\lambda_l} = 4\pi \int_0^\infty \frac{|g_l(q)|^2 q^2 \, dq}{q^2 + E_\mu}$$

so that

$$T_l(p, p', s) = \frac{g_l(p)g_l(p')}{s + E_\mu} \left(4\pi \int_0^\infty \frac{|g_l(q)|^2 q^2 \, dq}{(q^2 + E_\mu)(q^2 - s)}\right)^{-1}. \tag{4}$$

Comparing this with the known bound-state contribution to the T matrix

$$T_l(p, p', s) \xrightarrow[s \to -E_\mu]{} \frac{G_\mu(p)G_\mu(p')}{s + E_\mu},$$

where $G_\mu(p)$ is the *wavefunction* form factor, we see that $G_\mu(p) \equiv g_l(p)$ if the potential form factor was originally chosen to satisfy the condition

$$4\pi \int_0^\infty \frac{|g_l(q)|^2}{(q^2 + E_\mu)^2} q^2 \, dq = 1$$

which, since the wavefunction $\psi_\mu(p) = -G_\mu(p)/(p^2 + E_\mu)$, is equivalent to

$$4\pi \int_0^\infty |\psi_\mu(q)|^2 q^2 \, dq = 1.$$

Equation (4) may be rewritten in the form

$$T_l(p, p', s) = G_\mu(p)\mathcal{T}(s)G_\mu(p') \tag{5}$$

with

$$\mathcal{T}(s)^{-1} = (s + E_\mu) + 4\pi(s + E_\mu)^2 \int_0^\infty \frac{|\psi_\mu(q)|^2 q^2 \, dq}{q^2 - s}.$$

This provides a convenient parametrization of a system involving one bound state in which the off-shell behaviour is expressed in terms of the binding energy and the bound-state wavefunction and the potential no long appears explicitly.

The expression has obvious limitations in that the chosen form of the potential allows only one bound state in the particular partial wave considered. In molecular systems many bound states occur in each partial wave and it would be useful to extend this result to include an arbitrary number of bound states.

The obvious procedure is to write (dropping the suffix l)

$$\langle p|\hat{V}|p'\rangle = \sum \lambda_i g_i(p)g_i(p') \tag{6}$$

and one readily finds the solution (using an implied summation convention)

$$T(p, p', s) = g_i(p)\tau_{ij}(s)g_j(p') \tag{7}$$

(Newton 1966), with

$$\{\tau^{-1}(s)\}_{ij} = \frac{\delta_{ij}}{\lambda_i} + 4\pi \int_0^\infty \frac{g_i(q)g_j(q)q^2 \, \mathrm{d}q}{q^2 - s}.$$

This is a convenient mathematical result but it is not very useful from a physical point of view.

Suppose we want to simulate a system with N non-degenerate bound states with form factors $G_\mu(p)$ and binding energies E_μ, $\mu = 1, \dots, N$: the above expression involves the potential form factors $g_i(p)$ and the coupling constants λ_i, $i = 1, \dots, N$ whereas what is required is a transcription of equation (7) to a form analogous to equation (5) for the single separable potential. In addition the problem is not simply one of finding the N values λ_i that give the N bound states at $s = -E_\mu$ since for an arbitrary choice of potential form factors it is not clear that any choice of the λ_i will give N bound states. This 'input constraint' on the potential form factors is a reflection of the fact that ultimately the wavefunctions $\psi_\mu(p)$ are expressible in terms of the $g_i(p)$ and the wavefunctions themselves must satisfy orthogonality conditions; these conditions must therefore be contained in some complicated way in the initial choice of the potential form factors $g_i(p)$. This point is illustrated by the numerical calculations of Beregi *et al* (1970) who varied not only the coupling constants but also the functional form to obtain two bound states.

Although we have focused attention on the bound states it is again convenient to solve the scattering problem obtaining the off-shell T matrix $T(p, p', s)$. The bound states appear as poles of the T matrix at energies $s = -E_\mu$, $\mu = 1, \dots, N$.

3. The T matrix

In this section we assume that a superposition of N separable potentials (equation (6)) produces N bound states and then transform the T matrix (equation (7)) into an alternative form in which the potential no longer explicitly occurs. The final result is then a generalization of equation (5).

We know from the general property of the T matrix that

$$T(p, p', s) \xrightarrow[s \to -E_\mu]{} \frac{G_\mu(p)G_\mu(p')}{s + E_\mu} \tag{8}$$

where $G_\mu(p)$ is the wavefunction form factor. As far as equation (7) is concerned this factorization is achieved by

$$\tau_{jk}(s) \xrightarrow[s = -E_\mu]{} \frac{\alpha_j^\mu \alpha_k^\mu}{s + E_\mu}$$

with $G_\mu(p) = \Sigma \, \alpha_j^\mu g_j(p)$. We define an $N \times N$ matrix α by

$$\alpha_{\mu j} = \alpha_j^\mu$$

and its transpose $\tilde{\alpha}$ by

$$(\tilde{\alpha})_{j\mu} = \alpha_j^\mu.$$

Defining $\mathcal{T}^{-1} = \alpha \tau^{-1} \tilde{\alpha}$ we have

$$\mathcal{T}_{\mu\nu}^{-1} = \alpha_{\mu j} \tau_{jk}^{-1} \tilde{\alpha}_{k\nu}$$

$$= \alpha_j^\mu \left(\frac{\delta_{jk}}{\lambda_k} + 4\pi \int_0^\infty \frac{g_j(q) g_k(q) q^2 \, dq}{q^2 - s} \right) \alpha_k^\nu$$

$$= \sum \left(\alpha_j^\mu \frac{1}{\lambda_j} \alpha_j^\nu \right) + 4\pi \int_0^\infty \frac{G_\mu(q) G_\nu(q) q^2 \, dq}{q^2 - s}$$

$$= \beta_{\mu\nu} + I_{\mu\nu}(s). \tag{9}$$

Now

$$\tau^{-1} = \alpha^{-1} \mathcal{T}^{-1} \tilde{\alpha}^{-1}$$

so that

$$\tau = \tilde{\alpha} \mathcal{T} \alpha.$$

Hence equation (7) can be rewritten

$$T(p, p', s) = g_i(p) \alpha_i^\mu \mathcal{T}_{\mu\nu}(s) \alpha_j^\nu g_j(p') = G_\mu(p) \mathcal{T}_{\mu\nu}(s) G_\nu(p'). \tag{10}$$

which is equivalent to

$$(E_\mu - E_\rho) \int_0^\infty \psi_\mu(q)\psi_\rho(q)q^2 \, dq = 0. \tag{13}$$

This is guaranteed if the input wavefunctions are orthogonal.

Eliminating $\beta_{\mu\nu}$ from (9) gives

$$\mathcal{T}_{\mu\nu}^{-1}(s) = I_{\mu\nu}(s) - I_{\mu\nu}(-E_\mu) = (s + E_\mu)4\pi \int_0^\infty \frac{G_\mu(q)G_\nu(q)q^2 \, dq}{(q^2 + E_\mu)(q^2 - s)}.$$

This may be put into a more symmetric form using (13) supplemented by the normalization condition

$$4\pi \int_0^\infty |\psi_\mu(q)|^2 q^2 \, dq = 1$$

to give

$$\mathcal{T}_{\mu\nu}^{-1}(s) = \delta_{\mu\nu}(s + E_\mu) + 4\pi(s + E_\mu)(s + E_\nu) \int_0^\infty \frac{G_\mu(q)G_\nu(q)q^2 \, dq}{(q^2 + E_\mu)(q^2 + E_\nu)(q^2 - s)}$$

$$= \delta_{\mu\nu}(s + E_\mu) + 4\pi(s + E_\mu)(s + E_\nu) \int_0^\infty \frac{\psi_\mu(q)\psi_\nu(q)q^2 \, dq}{q^2 - s}. \tag{14}$$

Equation (14) inverted and substituted into (10) gives the generalization of equation (5) for the single separable potential.

4. Resonances

In the previous section we have considered only poles of the T matrix (in the energy variable) on the negative real axis of the first or physical sheet ($0 \le \arg s < 2\pi$). One might therefore expect that equations (10) and (14) satisfactorily give the off-shell behaviour for $s < 0$. It is however well known that there may be additional singularities on the unphysical sheet ($-\pi \le \arg s < 0$, $2\pi \le \arg s < 3\pi$) with poles occurring on the negative real axis or as complex conjugate pairs. Real poles on the unphysical sheet correspond to virtual states and the complex conjugate poles in the lower half-plane produce resonances if they approach the line $\arg s = 0$.

We therefore require a method of incorporating these poles as well as the bound state poles and Lovelace (1964a) has shown how to do this by investigating complex potentials related to the actual potential by the introduction of a phase factor.

Corresponding to the separable potential

$$V(p, p') = \sum \lambda_i g_i(p)g_i(p')$$

define a complex potential

$$V^\phi(p, p') = e^{3i\phi} \sum \lambda_i g_i(p e^{i\phi})g_i(p' e^{i\phi}) = e^{3i\phi} \sum_i \lambda_i g_i^\phi(p)g_i^\phi(p'). \tag{15}$$

In addition introduce a complex free-particle Hamiltonian

$$H_0^\phi(p, p') = p^2 e^{2i\phi} \frac{\delta(p - p')}{4\pi p^2}. \tag{16}$$

The spectrum of the free-particle Green function will now be on the line $\arg s = 2\phi$ rather than $\arg s = 0$ and the Green function will be analytic except for the cut along $\arg s = 2\phi$.

The T matrix is

$$T^\phi(p, p', s) = e^{3i\phi} g_i^\phi(p) \tau_{ij}^\phi(s) g_j^\phi(p')$$

with

$$(\tau^\phi(s)^{-1})_{ij} = \frac{\delta_{ij}}{\lambda_i} + e^{3i\phi} 4\pi \int \frac{g_i^\phi(q) g_j^\phi(q) q^2 \, dq}{q^2 e^{2i\phi} - s} = \frac{\delta_{ij}}{\lambda_i} + 4\pi \oint_{C_\phi} \frac{g_i(z) g_j(z) z^2 \, dz}{z^2 - s} \qquad (17)$$

where the integral now runs along the line $\arg z = \phi$; hence $\tau^\phi(s)$ is an analytic continuation of $\tau(s)$ into the region $\arg s < 0$ for $\phi < 0$ and $\arg s > 2\pi$ for $\phi > 0$. Lovelace has shown that the continuation holds for $|\phi| < \pi/2$. Thus if $2\phi < \min(\arg s_R)$ where s_R is the position of a pole in the region $-\pi < \arg s < 0$ the eigenstates of the complex Hamiltonian will correspond to the poles on $\arg s = \pi$ and those in the lower half of the unphysical sheet.

Labelling this set of singularities by their positions s_μ we see, following the arguments of the previous section, that

$$\tau_{ij}^\phi \xrightarrow[s \to s_\mu]{} \frac{\alpha_i^\mu(\phi) \alpha_j^\mu(\phi)}{s - s_\mu}$$

and

$$T^\phi(p, p', s) \xrightarrow[s \to s_\mu]{} G_\mu^\phi(p) \mathcal{T}_{\mu\nu}^\phi(s) G_\nu^\phi(p')$$

with

$$G_\mu^\phi(p) = \sum \alpha_i^\mu(\phi) g_i^\phi(p)$$

and

$$e^{3i\phi} (\mathcal{T}^\phi(s)^{-1})_{\mu\nu} = \sum \frac{\alpha_i^\mu(\phi) \alpha_i^\nu(\phi)}{\lambda_i} + e^{3i\phi} 4\pi \int \frac{G_\mu^\phi(q) G_\nu^\phi(q) q^2 \, dq}{q^2 e^{2i\phi} - s}. \qquad (18)$$

The eigenvalues s_μ correspond to (complex) energy values for which

$$(\tau^\phi(s_\mu)^{-1})_{ij} \alpha_j^\mu(\phi) = 0$$

has a non-trivial solution. For a given s_μ in the lower half-plane this will only be possible if $2\phi < 2\phi_\mu = \arg s_\mu$. However, for all ϕ such that $-\pi/2 < \phi < \phi_\mu$ the form of $\tau^\phi(s)^{-1}$ in equation (17) implies that

$$\alpha_i^\mu(\phi) = f_\mu(\phi) \alpha_i^\mu \qquad (19)$$

with $f_\mu(\phi)$ having modulus unity. This result follows by noting that $g_i(z)$ is a function of z^2 so that the integral in (17) can be converted into one running along the contour C_1 in figure 1. The singularities of the integrand are the poles at $z = \pm(s + i\epsilon)^{1/2}$ and the cuts and poles of $g_i(z)$ which lie on the imaginary axis. Contour integration (possible because the $g_i(p)$ must produce normalizable wavefunctions) shows that $\tau^\phi(s)^{-1}$ is independent of ϕ for $-\pi/2 < \phi < \phi_\mu$. Consequently the only dependence of $\alpha_i^\mu(\phi)$ upon ϕ is an overall phase factor and so (19) is the most general form.
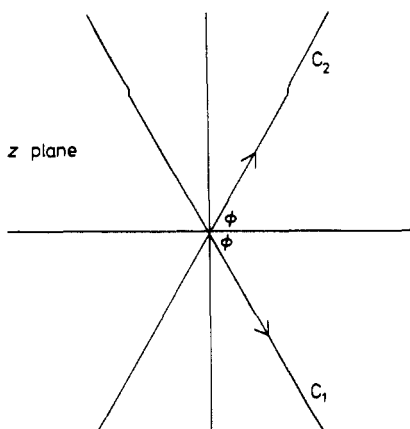
Figure 1. The complex momentum plane.

The $G_\mu^\phi(p)$ are the form factors associated with the eigenstates of the complex Hamiltonian and they are related to the eigenfunctions $\psi_\mu^\phi(p)$ by

$$\psi_\mu^\phi(p) = \frac{G_\mu^\phi(p)}{s_\mu - p^2 \, e^{2i\phi}}.$$

The symmetry of τ^{-1} gives identical left and right eigenfunctions and so the orthogonality condition is

$$4\pi \int \psi_\mu^\phi(p)\psi_\nu^\phi(p)p^2 \, \mathrm{d}p = \delta_{\mu\nu}.$$

Following the previous section we can now include both real and complex poles by choosing $2\phi < \min(\arg s_\mu)$ and writing

$$T^\phi(p, p', s) = G_\mu^\phi(p)\mathcal{T}_{\mu\nu}^\phi(s)G_\nu^\phi(p')$$

with

$$(\mathcal{T}^\phi(s)^{-1})_{\mu\nu} = \delta_{\mu\nu}(s - s_\mu) + 4\pi(s - s_\mu)(s - s_\nu) \int \frac{\psi_\mu^\phi(q)\psi_\nu^\phi(q)q^2 \, \mathrm{d}q}{q^2 \, e^{2i\phi} - s}. \quad (20)$$

The relevance of this to the real problem ($\phi = 0$) is given by Lovelace (1964a, b) who showed that (the phase dependence in the two references appears to be inconsistent, possibly due to a typographical error in Lovelace 1964a)

$$T^\phi(p, p', s) = e^{3i\phi} T(p \, e^{i\phi}, p' \, e^{i\phi}, s) \quad (21)$$

for

$$2\phi < \arg s < 2\pi \qquad \text{when } \phi \geq 0$$

and

$$0 < \arg s < 2\pi - 2|\phi| \qquad \text{when } \phi \leq 0.$$

For negative ϕ, the wavefunction $\psi_\mu^\phi(p)$ is only defined for $-\pi/2 < \phi < \phi_\mu$ but the form factor may be continued for $\phi > \phi_\mu$ according to

$$G_\mu^\phi(p) = e^{3i\theta/2} G_\mu^{\phi-\theta}(p \, e^{i\theta})$$

which shows, recalling

$$G_\mu^\phi(p) = \alpha_j^\mu(\phi)g_j(p\ e^{i\phi}) = f_\mu(\phi)\alpha_j^\mu g_j(p\ e^{i\phi}),$$

that

$$f_\mu(\phi) = e^{3i\phi/2}.$$

Continuing the form factors to $\phi = 0$ gives, using equations (20) and (21),

$$T(p, p', s) = G_\mu^0(p)\mathcal{T}_{\mu\nu}^\phi(s)G_\nu^0(p') \tag{22}$$

with

$$(\mathcal{T}^\phi(s)^{-1})_{\mu\nu} = \delta_{\mu\nu}(s - s_\mu) + 4\pi(s - s_\mu)(s - s_\nu)\oint_{C_\phi} \frac{G_\mu^0(z)G_\nu^0(z)z^2\ dz}{(z^2 - s_\mu)(z^2 - s_\nu)(z^2 - s)},$$

this result being valid, for negative ϕ, in the region $0 < \arg s < 2\pi - 2|\phi|$. A similar result holds, for ϕ positive, in the region $2\phi < \arg s < 2\pi$ with each s_μ replaced by s_μ^*.

This expression allows a parametrization of the two-body matrix in terms of an arbitrary number of bound states and resonances with form factors $G_\mu^0(p)$. The latter are chosen such that they satisfy the orthogonality condition along $\arg z = \phi < \min(\phi_\mu)$

$$4\pi\int_{C_\phi} \frac{G_\mu^0(z)G_\nu^0(z)z^2\ dz}{(z^2 - s_\mu)(z^2 - s_\nu)} = \delta_{\mu\nu}.$$

5. Unitarity

The off-shell unitarity condition is

$$T(p, p', s + i\epsilon) - T(p, p', s - i\epsilon) = -4\pi^2 kiT(p, k, s + i\epsilon)T(k, p', s - i\epsilon) \tag{23}$$

with $k = +\sqrt{s}$.

Using the representation (22) for $T(p, p', k^2 + i\epsilon)$ and the corresponding representation with $-\phi$ for $T(p, p', k^2 - i\epsilon)$ we have

$$T(p, p', k^2 + i\epsilon) - T(p, p', k^2 - i\epsilon)$$
$$= G_\mu^0(p)(\mathcal{T}_{\mu\nu}^\phi(k^2 + i\epsilon) - \mathcal{T}_{\mu\nu}^{-\phi}(k^2 - i\epsilon)G_\nu^0(p')$$
$$= G_\mu^0(p)\mathcal{T}_{\mu\rho}^\phi(k^2 + i\epsilon)D_{\rho\sigma}\mathcal{T}_{\sigma\nu}^{-\phi}(k^2 - i\epsilon)G_\nu^0(p') \tag{24}$$

with

$$D_{\rho\sigma} = (\mathcal{T}^{-\phi}(k^2 - i\epsilon)^{-1} - \mathcal{T}^\phi(k^2 + i\epsilon)^{-1})_{\rho\sigma}.$$

Using (18) we can write

$$D_{\rho\sigma} = 4\pi\alpha_i^\rho\left(\oint_{C_{-\phi}} \frac{g_i(z)g_j(z)z^2\ dz}{z^2 - k^2 + i\epsilon} - \oint_{C_\phi} \frac{g_i(z)g_j(z)z^2\ dz}{z^2 - k^2 - i\epsilon}\right)\alpha_j^\sigma$$
$$= 2\pi\alpha_i^\rho\left(\oint_{C_2} \frac{g_i(z)g_j(z)z^2\ dz}{z^2 - k^2 + i\epsilon} - \oint_{C_1} \frac{g_i(z)g_j(z)z^2\ dz}{z^2 - k^2 - i\epsilon}\right)\alpha_j^\sigma$$

where the integrals run along the contours C_1 and C_2 shown in figure 1. The singularities associated with the potential lie along the imaginary axis and their contributions to the two integrals cancel leaving

$$D_{\rho\sigma} = 2\pi(2\pi i)\alpha_i^\rho(-\tfrac{1}{2}kg_i(-k)g_j(-k) - \tfrac{1}{2}kg_i(k)g_j(k))\alpha_j^\sigma = -4\pi^2 ikG_\rho^0(k)G_\sigma^0(k).$$

Substitution of this into (24) demonstrates that the off-shell unitarity condition is satisfied.

6. Summary

The T matrix constructed in § 4 explicitly shows the contributions from the bound states, resonances and distant singularities in the complex energy plane. Consequently in situations where one is guessing a potential to describe a particular physical situation (22) provides a convenient alternative which leads directly to a T matrix whose off-shell behaviour is known explicitly.

A compact non-separable potential is one which can be approximated by

$$V(p, p') = \sum \lambda_n g_n(p) g_n(p')$$

with $\lambda_n \to 0$ as $n \to \infty$.

Such a potential will have a denumerable set of singularities but the form of the T matrix suggests that at any value of s it is the nearby singularities that control the off-shell behaviour. Consequently one can anticipate that the matrix $\mathcal{T}$ can be approximated by a finite matrix containing contributions from the nearest singularities and possibly a phenomenological contribution to represent the omitted distant singularities.

Such a parametrization will be particularly useful in tackling the three-body problem where it is essential to know the off-shell two-body T matrices. In particular for low-energy atom–molecule scattering below the break-up threshold important contributions occur from the 'unphysical scattering' of pairs of particles with negative two-body energies. This is the region dominated by the bound-state contributions and equations (10) and (14) are useful in describing this situation.

The T matrix forms derived here also have the advantage in that they allow a straightforward generalization of the Lovelace equations to include more than one bound state or resonance in each partial wave. The identity of the potential form factor and the wavefunction form factor when only one state is present in each partial wave simplifies the Lovelace equations; recasting the two-body T matrix in the form (22) provides a similar simplification in the general case.

Appendix

$$\mathcal{T}_{\mu\nu}(s) \xrightarrow[s\to -E_\rho]{} \frac{\delta_{\mu\nu}\delta_{\mu\rho}}{s+E_\rho} + A_{\mu\nu}.\tag{A.1}$$

Define: (a) $B_{\mu\nu}$ as the cofactor of $A_{\nu\mu}$ in the matrix A such that

$$\sum A_{\mu\gamma}B_{\gamma\nu} = \det A\delta_{\mu\nu}.\tag{A.2}$$

(b) The $(N-1)\times(N-1)$ matrix $\mathcal{A}$ obtained by omitting the ρth row and column from A.

(c) $C_{\mu\nu}$ as the cofactor of $A_{\nu\mu}$ in the matrix $\mathcal{A}$, i.e.

$$\sum_{\gamma\neq\rho} A_{\mu\gamma}C_{\gamma\nu} = \det \mathcal{A}\delta_{\mu\nu} = B_{\rho\rho}\delta_{\mu\nu}, \qquad \mu,\nu\neq\rho\tag{A.3}$$

$$\sum_{\gamma\neq\rho} A_{\rho\gamma}C_{\gamma\nu} = -B_{\rho\nu}\tag{A.4}$$

$$\sum_{\gamma\neq\rho} C_{\mu\gamma}A_{\gamma\rho} = -B_{\mu\rho}.\tag{A.5}$$

With these definitions the inverse of (A.1) is

$$(\mathcal{T}^{-1}(s))_{\mu\nu} \xrightarrow[s\to -E_\rho]{} \frac{s+E_\rho}{X}B_{\mu\nu} + \frac{1}{X}C'_{\mu\nu}\tag{A.6}$$

where

$$C'_{\mu\nu} = \begin{cases} C_{\mu\nu} & \text{if } \mu \text{ and } \nu \neq \rho \\ 0 & \mu \text{ or } \nu = \rho \end{cases}$$

and $X = B_{\rho\rho} + (s+E_\rho)\det A$.

Equations (A.2)–(A.5) can be used to show that (A.6) is in fact the inverse of (A.1) and the form of (A.6) shows that, at $s = -E_\rho$, $\mathcal{T}^{-1}(s)$ vanishes along the ρth row and column. This is the result used in § 3.

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